Dependent Random Graphs and Multiparty Pointer Jumping

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Abstract

We initiate a study of a relaxed version of the standard Erdős-Rényi random graph model, where each edge may depend on a few other edges. We call such graphs *dependent random graphs*. Our main result in this direction is a thorough understanding of the clique number of dependent random graphs. We also obtain bounds for the chromatic number. Surprisingly, many of the standard properties of random graphs also hold in this relaxed setting. We show that with high probability, a dependent random graph will contain a clique of size $\frac{(1-o(1))\log(n)}{\log(1/p)}$, and the chromatic number will be at most $\frac{n\log(1/(1-p))}{\log n}$. We expect these results to be of independent interest. As an application and second main result, we give a new communication protocol for the k-player Multiparty Pointer Jumping (MPJ_k) problem in the number-on-the-forehead (NOF) model. Multiparty Pointer Jumping is one of the canonical NOF communication problems, yet even for three players, its communication complexity is not well understood. Our protocol for MPJ_3 costs $O(n(\log\log n)/\log n)$ communication, improving on a bound from [8]. We extend our protocol to the non-Boolean pointer jumping problem $\widehat{\text{MPJ}}_k$, achieving an upper bound which is $o(n)$ for any $k \geqslant 4$ players. This is the first $o(n)$ protocol for $\widehat{\text{MPJ}}_k$ and improves on a bound of Damm, Jukna, and Sgall [10], which has stood for almost twenty years.

1 Introduction

Random Graphs. The study of random graphs revolves understanding the following distribution on graphs: Given n and p, define a distribution on n vertex graphs $G = (V, E)$ by placing each edge $(i, j) \in E$ **independently** with probability p. The first paper on this topic, authored by Erdős and Rényi [12], focused on connectivity of graphs. Later, Bollobás and Erdős [7] found the interesting result that almost every graph has a clique number of either r or $r + 1$, for some $r \approx \frac{2\log n}{\log 1/p}$. This remarkable concentration of measure result led to further investigations of these graphs. Then, Bollobás [5] solved the question of the chromatic number and showed that almost every graph has chromatic number $(1 + o(1))\frac{-n\log(1-p)}{2\log n}$. For more details, consult Bollobás [6] and Alon and Spencer [2].

We extend this model by allowing each edge to depend on up to d other edges. We make no a priori assumptions on *how* the edges depend on each other except that edges must be independent of all but at most d other edges. This defines a family of graph distributions. We initiate a study of dependent random graphs by considering the clique number and the chromatic number. As far as we know, this is the first work to systematically study such distributions. However, other relaxations of the standard random graph model have been studied. The most relevant for us is that of Alon and Nussboim [1], who study random graphs where edges are k-wise independent. [1] give tight bounds for several graph properties, including the clique number, the chromatic number, connectivity, and thresholds for the appearance of subgraphs. The bounds for k-wise independent graph properties are not as tight as the standard random graphs, but this is to be expected since k-wise independent random graphs are a family of distributions rather than a single

distribution. Our dependent random graphs similarly represent a family of graph distributions. However, dependent random graphs are generally not even almost k-wise independent, even for small values of d.

NOF Communication Complexity. As an application of our dependent random graphs, we study multiparty communication problems in the Number-On-The-Forehead (NOF) communication model defined by Chandra et al. [9]. In this model, there are k players $\text{PLR}_1, \cdots, \text{PLR}_k$ who wish to compute some function $f(x_1, \ldots, x_k)$ of their inputs using the minimal communication possible. Initially, players share a great deal of information: each PLR_i sees every input *except* x_i.[1] Note that a great deal of information is shared before communication begins; namely, all players except PLR_i see x_i. As a result, for many functions little communication is needed. Precisely how this shared information affects how much communication is needed is not currently well understood, even when limiting how players may communicate. We consider two well-studied models of communication. In the *one-way* communication model, players each send exactly one message in order (i.e., first PLR_1 sends his message, then PLR_2, etc.) In the *simultaneous-message* (or SM) model, each player simultaneously sends a single message to a referee, who processes the messages and outputs an answer. We use $\text{D}(f)$ and $\text{D}^{\|}(f)$ to denote the communication complexity of f in the one-way and simultaneous-message models respectively.

To date, no explicit function is known which requires a polynomial amount of communication for $k = O(\text{polylog}\, n)$ players in the SM model. Identifying such a function represents one of the biggest problems in communication complexity. Furthermore, a chain of results [19, 13, 4] showed that such a lower bound would place f outside of the complexity class ACC^0. ACC^0 lies at the frontier of our current understanding of circuit complexity, and until the recent work of Williams [18] it wasn't even known that $\mathsf{NEXP} \nsubseteq \mathsf{ACC}^0$. The Multipary Pointer Jumping problem is widely conjectured to require enough communication to place it outside of ACC^0. This motivates our study.

The Pointer Jumping Problem. There are many variants of the pointer jumping problem. Here, we study two: a Boolean version MPJ_k^n, and a non-Boolean version $\widehat{\text{MPJ}}_k^n$. (From now on, we suppress the n to ease notation). We shall formally define these problems in Section 2, but for now, each may be described as problems on a directed graph that has $k+1$ layers of vertices $L_0, \ldots, L_k$. The first layer L_0 contains a single vertex s_0, and layers $L_1, \ldots, L_{k-1}$ contain n vertices each. In the Boolean version, L_k contains two vertices, while in the non-Boolean version L_k contains n vertices. For inputs, each vertex in each layer except L_k has a single directed edge pointing to some vertex in the next layer. The output is the the unique vertex in L_k reachable from s_0; i.e., the vertex reached by starting at s_0 and "following the pointers" to the kth layer. Note that the output is a single bit for MPJ_k and a $\log n$-bit string for $\widehat{\text{MPJ}}_k$. To make this into a communication game, we place on PLR_i's forehead all edges from vertices in L_{i-1} to vertices in L_i. If players speak in any order except $\text{PLR}_1, \cdots, \text{PLR}_k$, there is an easy $O(\log n)$-bit protocol for MPJ_k.

This problem was first studied by Wigderson,[2] who gave an $\Omega(\sqrt{n})$ lower bound for MPJ_3. This was later extended by Viola and Wigderson [17], who showed that MPJ_k requires $\tilde{\Omega}(n^{1/(k-1)})$ communication, even under randomized communication. On the upper-bounds side, Pudlak et al. [16] showed a protocol for MPJ_3 that uses only $O\left(n(\log\log n)/\log n\right)$ communication, but only works when the input on PLR_2's forehead is a permutation. Damm et al. [10] show that $D(\widehat{\text{MPJ}}_3) = O(n \log\log n)$ and $D(\widehat{\text{MPJ}}_k) = O(n \log^{(k-1)} n)$, where $\log^{(r)} n$ is the rth iterated log of n. Building on [16], Brody and Chakrabarti [8] showed $D(\text{MPJ}_3) = O\left(n\sqrt{(\log\log n)/\log n}\right)$; they give marginal improvements for MPJ_k for $k > 3$. Despite the attention

[1] Imagine x_i being written on PLR_i's forehead. Then, PLR_i sees inputs on other players' foreheads, but not his own.

[2] This was unpublished, but an exposition appears in [3].

devoted to this problem, the upper and lower bounds remain far apart, even for $k = 3$ players, where $D(\text{MPJ}_3) = \Omega(\sqrt{n})$ and $D(\text{MPJ}_3) = O(n\sqrt{(\log\log n)/\log n})$. For this reason, in this work we focus on MPJ_k and $\widehat{\text{MPJ}}_k$ for small values of k. We strongly believe that fully understanding the communication complexity of MPJ_3 will shed light on the general problem as well.

1.1 Our Results

We give two collections of results: one for dependent random graphs, and the other for the communication complexity of MPJ_k and $\widehat{\text{MPJ}}_k$. For our work on dependent random graphs, we focus on the clique number and on the chromatic number. The clique number of a graph G, denoted $\text{clique}(G)$, is the size of the largest clique; the chromatic number $\chi(G)$ is the number of colors needed to color the vertices such that the endpoints of each edge have different colors. We use e.g. $\text{clique}(G_d(n,p))$ to refer to $\text{clique}(G)$ for some $G \sim G_d(n,p)$. We achieve upper and lower bounds for each graph property. Say that a graph property P holds **almost surely (a.s.)** if it holds with probability approaching 1 as n approaches ∞ i.e. if P holds with probability $1 - o(1)$.

Our strongest results[3] give a lower bound for $\text{clique}(G_d(n,p))$ and an upper bound for $\chi(G_d(n,p))$.

Theorem 1. *If $0 < p < 1/4$ and $d/p << \sqrt{n}$, then $G_d(n,p)$ almost surely has a clique of size $\Omega\left(\frac{\log n}{\log 1/p}\right)$.*

Theorem 2. *If $3/4 < p < 1$ and $d = n^{o(1)}$ then almost surely $\chi(G_d(n,p)) \leqslant (1+\varepsilon)\frac{-n\log(1-p)}{\log n}$.*

These bounds nearly match similar results for Erdős-Rényi random graphs. Our bounds on the other side are not as tight.

Theorem 3. *If $0 < p < 1$ and $d \leqslant n/\log^2 n$, then almost surely $\text{clique}(G_d(n,p)) \leqslant d\log n$.*

Theorem 4. *If $0 < p < 1$ and $d \leqslant n/\log^2 n$, then almost surely $\chi(G_d(n,p)) \geqslant n/(d\log n)$.*

For large values of d, there are wide gaps in the upper and lower bounds of clique number and chromatic number. Are these gaps necessary? The existing bounds for random graphs show that Theorems 1 and 2 are close to optimal. Our next result witnesses the tightness for $\text{clique}(G_d(n,p))$.

Lemma 5. *For any $d = o(n)$ and any $0 < p < 1$*

1. *there are d-dependent random graphs that almost surely contain cliques of size $\Omega(d)$.*
2. *there are d-dependent random graphs that almost surely contain cliques of size $\Omega(\sqrt{d}\log n)$.*

This result shows that Theorem 3 is also close to optimal. It also demonstrates that tight concentration of measure does not generally hold for dependent random graphs, even for small values of d. Nevertheless, we expect that for many specific dependent random graphs, tight concentration of measure results will hold. Finally, we give two simple constructions which show that with too much dependence, very little can be said about $\text{clique}(G_d(n,p))$.

Lemma 6. *For any $d \geqslant 2n$, the following statements hold.*

1. *For any $0 < p < 1$, there exists a d-dependent random graph $G_d(n,p)$ that is bipartite with certainty.*
2. *For any $1/2 \leqslant p < 1$, there exists a d-dependent random graph $G_d(n,p)$ that contains a clique of size $n/2$ with certainty.*

[3] Our choice of p is motivated by what was needed to obtain the communication complexity bounds for MPJ_k. We suspect that tweaking our technical lemmas will give bounds for any constant p.

Results for Multiparty Pointer Jumping. Our main NOF communication complexity result is a new protocol for MPJ_3.

Theorem 7. $\mathrm{D}(\text{MPJ}_3) = O(n(\log\log n)/\log n)$.

This is the first improvement in the communication complexity of MPJ since the work of Brody and Chakrabarti [8]. Next, we use this protocol to get new bounds for the non-Boolean version.

Theorem 8. $\mathrm{D}(\widehat{\text{MPJ}}_4) = O\left(n\frac{(\log\log n)^2}{\log n}\right)$.

Our protocol for $\widehat{\text{MPJ}}_4$ is the first sublinear-cost protocol for $\widehat{\text{MPJ}}_k$ for any value of k and improves on the protocol of Damm et al. [10] which has stood for nearly twenty years. Our last pointer jumping results give upper bounds in the SM setting. First we show how to convert our protocol from Theorem 7 to a simultaneous messages protocol.

Lemma 9. $\mathrm{D}^{\|}(\text{MPJ}_3) = O\left(n\frac{\log\log n}{\log n}\right)$.

Note that to solve $\widehat{\text{MPJ}}_3$, players can compute each bit of $f_3(f_2(i))$ using an MPJ_3 protocol. By running $\log n$ instances in parallel, players compute all of $\widehat{\text{MPJ}}_3(i, f_2, f_3)$. Thus, we get the following bound for $\widehat{\text{MPJ}}_3$.

Corollary 10. $\mathrm{D}^{\|}(\widehat{\text{MPJ}}_3) = O(n\log\log n)$.

This matches the bound from [10] but holds in the more restrictive SM setting.

1.2 Obtaining Bounds for Dependent Random Graph Properties

In this subsection, we describe the technical hook we obtained to prove our bounds for Theorems 1 and 2. A key piece of intuition is that when looking at only small subgraphs of $G \sim G_d(n,p)$, the subgraph usually *looks* like $G(n,p)$. This intution is formalized in the following definition and lemma.

Definition 1.1. *Given a dependent random graph $G_d(n,p)$, call a subset of vertices $S \subseteq V$* UNCORRELATED *if any two edges in the subgraph induced by S are independent.*

Lemma 11. *Suppose d and k are integers such that $dk^3 \leqslant n$. Fix any d-dependent graph $G_d(n,p)$, and let S be a set of k vertices uniformly chosen from V. Then, we have*

$$\Pr[S \textit{ is } \text{UNCORRELATED}] \geqslant 1 - \frac{3dk^3}{2n}\ .$$

At first glance, it might appear like we are now able to appeal to the existing arguments for obtaining bounds for $\mathrm{clique}(G(n,p))$ and then $\chi(G(n,p))$. Unfortuantely, this is not the case—while most potential k-cliques are UNCORRELATED, allowing correlation between edges drives up the *variance*. In effect, we might expect to have roughly the same number of k-cliques, but these cliques bunch together. Nevertheless, we are able to show that when d is small enough, these cliques don't bunch up too much. Appropriately bounding the variance is the most technically involved hurdle in this work, and is necessary to obtain both the upper bound on the chromatic number, and the effecient pointer jumping protocol.

1.3 Roadmap

The rest of the paper is organized as follows. In Section 2 we specify some notation, give formal definitions for the problems and models we consider, and provide some technical lemmas on probability which we'll need in later sections. We develop our results for dependent random graphs in Section 3, deferring some technical lemmas to Section 5. We present main result on Multiparty Pointer Jumping in Section 4, deferring the secondary MPJ_k results to Section 6. In Section 7 we prove Lemmas 5 and 6.

2 Preliminaries and Notation

We use $[n]$ to denote the set $\{1, \ldots, n\}$, N to denote $\binom{n}{2}$, and $\exp(z)$ to denote e^z. For a string $x \in \{0,1\}^n$, let $x[j]$ denote the jth bit of x. For a sequence of random variables $X_0, X_1, \ldots$, we use $\mathbf{X}_i$ to denote the subsequence $X_0, \ldots, X_i$. For a graph $G = (V, E)$, $\bar{G}$ denotes the complement of G. Given sets $A \subset B \subset V$, we use $B \backslash A$ to denote the set of edges $\{(u, v) : u, v \in B \text{ and } \{u, v\} \nsubseteq A\}$.

For a communication problem, we refer to players as $\text{PLR}_1, \ldots, \text{PLR}_k$. When $k = 3$, we anthropomorphize players as Alice, Bob, and Carol. Our communication complexity measures were defined in Section 1; for an in-depth development of communication complexity, consult the excellent standard textbook of Kushilevitz and Nisan [15].

2.1 Probability Theory and Random Graphs

Next, we formalize our notion of dependent random graphs and describe the tools we use to bound $\text{clique}(G_d(n,p))$.

Definition 2.1 ([11], Definition 5.3). *A sequence of random variables $Y_0, Y_1, \ldots, Y_n$ is a* martingale *with respect to another sequence $X_0, X_1, \ldots, X_n$ if for all $i \geqslant 0$ we have*

$$Y_i = g_i(\boldsymbol{X}_i)$$

for some functions $\{g_i\}$ and, for all $i \geqslant 1$ we have

$$E[Y_i | \boldsymbol{X}_{i-1}] = Y_{i-1} \, .$$

Theorem 12 (Azuma's Inequality). *Let $Y_0, \ldots, Y_n$ be a martingale with respect to $X_0, \ldots, X_n$ such that $a_i \leqslant Y_i - Y_{i-1} \leqslant b_i$ for all $i \geqslant 1$. Then*

$$\Pr[Y_n > Y_0 + t], \Pr[Y_n < Y_0 - t] \leqslant \exp\left(-\frac{2t^2}{\sum_i (b_i - a_i)^2}\right) \, .$$

Of particular relevance for our work is the *edge-exposure martingale*. Let G be a random graph. Arbitrarily order possible edges of the graph $e_1, \ldots, e_N$, and let X_i be the indicator variable for the event that $e_i \in G$. Let $f : \binom{n}{2} \to \mathbb{R}$ be any function on the edge set, and let $Y_i := E[f(X_1, \ldots, X_N) | \mathbf{X}_i]$. It is easy to verify that for any f, $E[Y_i | \mathbf{X}_{<i}] = Y_{i-1}$, and therefore $\{Y_i\}$ are a matingale with respect to $\{X_i\}$. We say that $\{Y_i\}$ is the edge-exposure martingale for G.

It is worth noting that martingales make no assumptions about the independence of $\{X_i\}$. We'll use martingales on graph distributions where each edges may depend on a small number of other edges. This notion of *local dependency* is formalized below.

A *dependency graph* for a set of random variables $X = \{X_1, \ldots, X_N\}$ is a graph H on $[N]$ such that for all i, X_i is independent of $\{X_j : (i,j) \notin H\}$. We say that a set of variables X is d-**locally dependent** if there exists a dependency graph for X where each vertex has degree at most d.

The following lemma of Janson [14] (rephrased in our notation) bounds the probability that the sum of a series of random bits deviates far from its expected value, when the random bits have limited dependence.

Lemma 13. *[14] Let $X = \{X_i\}_{i\in[N]}$ be a d-locally dependent set of identically distributed binary variables, and let $Y = \sum_{i\in[N]} X_i$. Then, for any t we have*

$$\Pr[|Y - \mathbb{E}[Y]| \geqslant t] \leqslant e^{\frac{-2t^2}{(d+1)N}} .$$

For more details and results on probability and concentration of measure, consult the textbook of Dubhashi and Panconesi [11].

Definition 2.2. *A distribution $G_d(n,p)$ is* d-dependent *if each edge is placed in the graph with probability p, and furthermore that the set of edges are d-locally dependent.*

Note that taking $d = 0$ gives the standard Erdős-Rényi graph model. As with k-wise independent random graphs, d-dependent random graphs are actually a family of graph distributions. We make no assumptions on the underlying distribution beyond the fact that each edge depends on at most d other edges. We use $G_d(n,p)$ to denote an arbitrary dependent random graph.

A clique in a graph $G = (V,E)$ is a set of vertices S such that the subgraph induced on S is complete. Similarly, an independent set T is a set of vertices whose induced subgraph is empty. A *clique cover* of G is a partition of V into cliques. We let $\mathrm{clique}(G)$ denote the size of the largest clique in G. Let $\chi(G)$ denote the chromatic number of G; i.e., the minimum number of colors needed to color the vertex set such that no two adjacent vertices are colored the same. Note that $\chi(G)$ is the size of the smallest clique cover of $\bar{G}$.

2.2 Multiparty Pointer Jumping

Finally, we formally define the Boolean Multiparty Pointer Jumping function. Let $i \in [n]$, and let $f_2, \ldots, f_k : [n]^n$, be functions from $[n]$ to $[n]$. Let $x \in \{0,1\}^n$. We define the k-player pointer jumping function $\textsc{mpj}_k^n : [n] \times ([n]^n)^{k-2} \times \{0,1\}^n$ recursively as follows:

$$\begin{aligned} \textsc{mpj}_3^n(i, f_2, x) &:= x[f_2(i)] , \\ \textsc{mpj}_k^n(i, f_2, \ldots, f_{k-1}, x) &:= \textsc{mpj}_{k-1}^n(f_2(i), f_3, \ldots, f_{k-1}, x) . \end{aligned}$$

The non-Boolean version $\widehat{\textsc{mpj}}_k^n : [n] \times ([n]^n)^{k-1}$ is defined similarly recursively:

$$\begin{aligned} \widehat{\textsc{mpj}}_3^n(i, f_2, f_3) &:= f_3(f_2(i)) , \\ \widehat{\textsc{mpj}}_k^n(i, f_2, \ldots, f_k) &:= \widehat{\textsc{mpj}}_{k-1}^n(f_2(i), f_3, \ldots, f_k) . \end{aligned}$$

Henceforth, we drop the superscript n to ease notation. Each problem is turned into a communication game in the natural way. $\textsc{plr}_1$ is given i; for each $2 \leqslant j < k$, $\textsc{plr}_j$ receives f_j, and $\textsc{plr}_k$ receives x. Players must communicate to output $\textsc{mpj}_k(i, f_2, \ldots, f_{k-1}, x)$.

3 Dependent Random Graphs

In this section, we prove our main results regarding dependent random graphs, namely that with high probability they contain a large clique, and with high probability the chromatic number is not too large. The two theorems are formally stated below.

Theorem 14. *(Formal Restatement of Theorem 1) For all $0 < \varepsilon < 1/4$ there exists n_0 such that*

$$\Pr[\text{clique}(G_d(n,p)) > k] > 1 - \exp(-n^{1+\varepsilon}) ,$$

for all $n \geqslant n_0$, for all $n^{-\varepsilon/4} < p < \frac{1}{4}$ and for all d, k such that $k \leqslant \frac{\log(n/(2d\log^3 n))}{\log(1/p)}$ and $d/p \leqslant n^{1/2-\varepsilon}$.

This theorem shows $\text{clique}(G_d(n,p)) = \Omega\left(\frac{\log n}{\log 1/p}\right)$ with high probability, as long as d/p is bounded away from $\sqrt{n}$. Furthermore, when $d = n^{o(1)}$, $\text{clique}(G_d(n,p)) \geqslant (1-\varepsilon)\frac{\log n}{\log(1/p)}$ with high probability.

Proof. This proof follows the classic technique of Bollobás [5], modified to handle dependent random graphs. We need to show that $G_d(n,p)$ contains clique of size k. To that end, let Y be the largest number of *edge-disjoint* UNCORRELATED k-cliques. First, we give a lower bound on $E[Y]$; we defer its proof to Section 5.

Lemma 15. $\mathbb{E}[Y] \geqslant \frac{n^2 p}{19k^5}$.

Now, we use the edge-exposure martingale on $G_d(n,p)$ to show that with high probability, Y does not stray far from it's expectation. Let $Y_0, Y_1, \cdots Y_N$, be the edge exposure martingale on $G_d(n,p)$. Recall that $Y_0 = E[Y], Y_N = Y$, and $Y_i = E[Y|\mathbf{X}_i]$. In a standard random graph model where all edges are independently placed in G, it is easy to see that conditioning on whether or not an edge is in the graph changes the expected number of *edge-disjoint* UNCORRELATED k-cliques by at most one. This no longer holds when edges are dependent. However, if the graph distribution is d-dependent, then conditioning on X_i changes the expected number of *edge-disjoint* UNCORRELATED k-cliques by at most d. Therefore, $|Y_{i+1} - Y_i| \leqslant d$. Then, by Azuma's inequality, Lemma 15, and our assumption that $d/p \leqslant n^{1/2-\varepsilon}$, we have

$$\begin{aligned}
\Pr[Y = 0] &\leqslant \Pr[Y - \mathbb{E}[Y] \leqslant -\mathbb{E}[Y]] \\
&\leqslant \exp\left(\frac{-\mathbb{E}[Y]^2}{2Nd^2}\right) \\
&= \exp\left(-\frac{n^2p^2}{19^2d^2k^{10}}(1+o(1))\right) \\
&\leqslant \exp(-n^{1+\varepsilon}) .
\end{aligned}$$

Thus, it follows that $G_d(n,p)$ contains an UNCORRELATED k-clique with probability at least $1-\exp(-n^{1+\varepsilon})$. Since every UNCORRELATED clique is still a clique, it is clear that

$$\Pr[\text{clique}(G_d(n,p)) \geqslant k] \geqslant 1 - \exp(-n^{1+\varepsilon}) .$$

□

Next, we use the lower bound on $\text{clique}(G_d(n,p))$ to obtain an upper bound on $\chi(G_d(n,p))$.

Theorem 16. *For all $0 < \varepsilon < 1/8$ there exists n_0 such that*

$$\Pr\left[\chi(G_d(n,q)) < (1+4\varepsilon)\frac{-n\log(1-q)}{\log n}\right] > 1 - \exp(n^{1+\varepsilon}) ,$$

for all $3/4 < q < 1 - n^{-\varepsilon/4}$, all $d \leqslant n^{o(1)}$, and all $n \geqslant n_0$.

Proof. This follows a greedy coloring approach similar to [5, 16], but adapted to dependent random graphs. Set $m = \frac{n}{\log^2 n}$, $\varepsilon' = 2\varepsilon$, and $p = 1 - q$. Let $\mathcal{E}$ be the event that every induced subgraph H of $G_d(n, q)$ with m vertices has an independent set of size at least $k := (1 - \varepsilon')\frac{\log m}{-\log(1-q)}$. Independent sets in $G_d(n, q)$ correspond to cliques in the complement graph $\overline{G_d(n, q)}$, which is distributed identically to $G_d(n, p)$. Thus, we're able to leverage Theorem 14 to bound $\Pr[\mathcal{E}]$. In particular, since $d \leqslant n^{o(1)} \leqslant m^{o(1)}$,[4] by Theorem 14 and a union bound we have

$$\Pr[\mathcal{E}] > 1 - \binom{n}{m} \exp(-n^{1+\varepsilon'}) > 1 - \exp\left(\frac{n}{\log n} - n^{1+\varepsilon'}\right) > 1 - \exp(-n^{1+\varepsilon}) \,.$$

Now, assume $\mathcal{E}$ holds. We iteratively construct a coloring for $G_d(n, q)$. Start with each vertex uncolored. Repeat the following process as long as more than m uncolored vertices remain: Select m uncolored vertices. From their induced subgraph, identify an independent set I of size at least k. Then, color each vertex in I using a new color. When at most m uncolored vertices remain, color each remaining vertex using a different color. Since two vertices share the same color only if they are in an independent set, it's clear this is a valid coloring. More over, for each color in the first phase, we color at least $k > (1 - \varepsilon')\frac{\log m}{-\log p} > (1 - (3/2)\varepsilon)\frac{\log n}{-\log p}$ vertices. Hence, the overall number of colors used is at most

$$\frac{n - m}{(1 - (3/2)\varepsilon')(\log n)/(-\log(1 - q))} + m \leqslant (1 + 4\varepsilon)\frac{-n \log(1 - q)}{\log n} \,.$$

Therefore, $\chi(G_d(n, q)) \leqslant (1 + 4\varepsilon)\frac{-n\log(1-q)}{\log n}$ as long as $\mathcal{E}$ holds. This completes the proof. □

Finally, we give an upper bound on $\mathrm{clique}(G_d(n, p))$ and a lower bound on $\chi(G_d(n, p))$, which follow directly from Lemma 13.

Theorem 17. *For all $0 < p < 1$ and $d \leqslant n/\log^2 n$, almost surely* $\mathrm{clique}(G_d(n, p)) = O(d \log n)$.

Proof. Let $G \sim G_d(n, p)$, and fix some constant c to be determined later. For a set of vertices $S \subseteq V$ of size $|S| = cd \log n$, let BAD_S denote the event that S is a clique, and let $BAD := \bigvee_S BAD_S$. Note that there are $\binom{n}{cd\log n} \leqslant exp(cd \log^2 n)$ such events. Since G is d-dependent and $S \subset V$, then the subgraph induced by S is also d-dependent. Now, define $z := \binom{cd\log n}{2}$ and let $X_1, \ldots, X_z$ be indicator variables for the edges in the subgraph induced by S. Finally, let $Y := \sum_i X_i$. Then, $E[Y] = pz$, and BAD_S amounts to having $Y = z$. By Lemma 13,

$$\Pr[BAD_S] = \Pr[Y = z] = \Pr[Y - E[Y] \geqslant z(1-p)] \leqslant \exp\left(-\frac{2z^2(1-p)^2}{(d+1)z}\right) = \exp\left(-\frac{2z(1-p)^2}{d+1}\right) \,.$$

Choosing $c = 1/(1-p)^2$ and using a union bound yields

$$\begin{aligned}
\Pr[BAD] &\leqslant \binom{n}{z} \Pr[BAD_S] \leqslant \exp\left(cd\log^2 n - \frac{2(1-p)^2}{d+1}(cd\log n)^2\right) \\
&= \exp\left(cd\log^2 n(1 - 2c(1-p)^2)\right) \\
&< \exp(-\Omega(d\log^2 n)) \,,
\end{aligned}$$

Thus, almost surely $G_d(n, p)$ has no clique of size $\geqslant cd \log n$. □

[4] note that $n^\delta = m^{\delta'}$, where $\delta' = \delta\frac{\log n}{\log n - 2\log\log n}$. If $\delta = o(1)$ then $\delta' = o(1)$ as well.

Our lower bound on $\chi(G_d(n,p))$ follows as a direct corollary, since any independent set in $G_d(n,p)$ is a clique in the complement graph $\overline{G_d(n,p)}$, which is also d-dependent.

Corollary 18. *If $0 < p < 1$ and $d \leqslant n/\log^2 n$, then almost surely $\chi(G_d(n,p)) \geqslant n/(d \log n)$.*

4 A New Protocol for MPJ_3

Below, we describe a family of MPJ$_3$ protocols $\{\mathcal{P}_H\}$ parameterized by a bipartite graph $H = (A \cup B, E)$ with $|A| = |B| = n$. In each protocol $\mathcal{P}_H$, Alice and Bob each independently send a single message to Carol, who must take the messages and the input she sees and output MPJ$_3(i, f, x)$. Bob's communication in each protocol is simple: given i, he sends x_j for each j such that $(i,j) \in H$. Alice's message is more involved. Given H and f, she partitions $[n]$ into *clusters*. For each cluster in the partition, she sends the XOR of the bits for x. (e.g. if one cluster is $\{1,3,5\}$, then Alice would send $x[1] \oplus x[3] \oplus x[5]$) This partition of $[n]$ into clusters is carefully chosen and depends on H and f. Crucially, it is possible to make this partition so that for any inputs i, f, Bob sends $x[j]$ for each j in the cluster containing $f(i)$, except for possibly $x[f(i)]$ itself. We formalize this clustering below. Thus, Carol can compute $x[f(i)]$ by taking the relevant cluster from Alice's message and "XOR-ing out" the irrelevant bits using portions of Bob's message.

Each protocol $\mathcal{P}_H$ will correctly compute MPJ$_3(i, f, x)$; we then use the probabilitic method to show that there exists a graph H such that $\mathcal{P}_H$ is *efficient*. At the heart of this probabilistic analysis is a bound on the chromatic number of a dependent random graph. For functions with large preimages, this dependency becomes too great to handle.

Definition 4.1. *A function $f : [n] \to [n]$ is* d-limited *if $|f^{-1}(j)| \leqslant d$ for all $j \in [n]$.*

We end up with a protocol $\mathcal{P}_H$ that is efficient for all inputs (i, f, x) as long as f is d-limited ($d \approx \log n$ suffices); later, we generalize $\mathcal{P}_H$ to work for all inputs.

Remark 1. *This construction is inspired by the construction of Pudlák et al. [16], who gave a protocol for* MPJ$_3$ *that works in the special case that the middle layer is a* permutation π *instead of a general function f. They also use the probabilistic method to show that one $\mathcal{P}_H$ must be efficient. The probablistic method argument in our case depends on the chromatic number of a dependent random graph; the analysis of the permutation-based protocol in [16] relied on the chromatic number of the standard random graph $G(n,p)$.*

Description of $\mathcal{P}_H$. Let $H = (A \cup B, E)$ be a bipartite graph with $|A| = |B| = n$. Given H and f, define a graph $G_{f,H}$ by placing $(i,j) \in G_{f,H}$ if and only if both $(i, f(j))$ and $(j, f(i))$ are in H. Let $C_1, \ldots, C_k$ be a clique cover of $G_{f,H}$, and for each $1 \leqslant \ell \leqslant k$, let $S_\ell := \{f(j) : j \in C_\ell\}$.

The protocol $\mathcal{P}_H$ proceeds as follows. Given f and x, Alice constructs $G_{f,H}$. For each clique C_ℓ, Alice sends $b_\ell := \bigoplus_{j \in S_\ell} x[j]$. Bob, given i and x, sends $x[j]$ for all $(i,j) \in H$. We claim these messages enable Carol to recover MPJ$_3(i, f, x)$. Indeed, given i and f, Carol computes $G_{f,H}$. Let C be the clique in the clique cover of $G_{f,H}$ containing i, and let $S := \{f(j) : j \in C\}$ and $b := \bigoplus_{j \in S} x[j]$. Note that Alice sends b. Also note that for any $j \neq i \in C$, there is an edge $(i,j) \in G_{f,H}$. By construction, this means that $(i, f(j)) \in H$, so Bob sends $x[f(j)]$. Thus, Carol computes $x[f(i)]$ by taking b (which Alice sends) and "XOR-ing out" $x[f(j)]$ for any $j \neq i \in C$. In this way, $\mathcal{P}_H$ computes MPJ$_3$.

While $\mathcal{P}_H$ computes MPJ$_3$, it might not do so in a communication-efficient manner. The following lemma shows that there is an efficient protocol whenever f has small preimages.

Lemma 19. *For any $d \leqslant n^{o(1)}$, there exists a bipartite graph H such that for all $i \in [n], x \in \{0,1\}^n$, and all d-limited functions f, we have*

$$\mathrm{cost}(\mathcal{P}_H) = O\left(n\frac{\log\log n}{\log n}\right) .$$

Before proving Lemma 19, let us see how this gives the general upper bound.

Theorem 20 (Restatement of Theorem 7). $\mathrm{D}(\textsc{mpj}_3) = O(n(\log\log n)/\log n)$.

Proof. Fix $d = \log n$ and let $\mathcal{P}_H$ be the protocol guaranteed by Lemma 19. We construct a general protocol $\mathcal{P}$ for $\textsc{mpj}_3$ as follows. Given f, Alice and Carol select a d-limited function g such that $g(j) = f(j)$ for all j such that $|f^{-1}(f(j))| \leqslant d$. Note that Alice and Carol can do this without communication, by selecting (say) the lexicographically least such g. On input (i, f, x), Alice sends the message she would have sent in $\mathcal{P}_H$ on input (i, g, x), along with $x[j]$ for all j with large preimages. Bob merely sends the message he would have sent in $\mathcal{P}_H$. If the preimage of $f(i)$ is large, then Carol recovers $x[f(i)]$ directly from the second part of Alice's message. Otherwise, Carol computes $\textsc{mpj}_3(i, g, x)$ using $\mathcal{P}_H$. Since $f(i)$ has a small preimage, we know that $x[g(i)] = x[f(i)] = \textsc{mpj}_3(i, f, x)$, so in either case Carol recovers $\textsc{mpj}_3(i, f, x)$.

The communication cost of $\mathcal{P}$ is the cost of $\mathcal{P}_H$, plus one bit for each j with preimage $|f^{-1}(j)| > d$. There are at most n/d such j. With $d = \log n$ and using Lemma 19, the cost of $\mathcal{P}$ is

$$\mathrm{cost}(\mathcal{P}) \leqslant \mathrm{cost}(\mathcal{P}_H) + n/d = O(n(\log\log n)/\log n) + O(n/\log n) = O(n(\log\log n)/\log n) .$$

□

Proof of Lemma 19. We use the Probabilitstic Method. Place each edge in H independently with probability $p = \Theta\left(\frac{\log\log n}{\log n}\right)$. Now, for any d-limited function f, consider the graph $G_{f,H}$. Each edge (i, j) is in $G_{f,H}$ with probability p^2, but the edges are not independent. However, we claim that if f is d-limited, then $G_{f,H}$ is $(2d-2)$-dependent. To see this, note that (i, j) is in $G_{f,H}$ if both $(i, f(j))$ and $(j, f(i))$ are in H. Therefore, (i, j) is dependent on (i) any edge (i, j') such that $f(j') = f(j)$, and (ii) any edge (i', j) such that $f(i) = f(i')$. Since f is d-limited, there are at most $d-1$ choices each for i' and j'. Thus, each edge depends on at most $2d-2$ other edges, and $G_{f,H}$ is $(2d-2)$-dependent.

In $\mathcal{P}_H$, Alice sends one bit per clique in the clique cover of $G_{f,H}$. Bob sends one bit for each neighbor of i in H. Thus, we'd like a graph H such that every $i \in [n]$ has a few neighbors and every d-limited function f has a small clique cover.

Let BAD_i denote the event that i has more than $2pn$ neighbors in H. By a standard Chernoff bound argument, $\Pr[BAD_i] \leqslant \exp(-np^2/2)$. Next, let BAD_f be the event that at least $(1+\varepsilon)\frac{-n\log(p^2)}{\log n}$ cliques are needed to cover the vertices in $G_{f,H}$. Note that any clique in $G_{f,H}$ is an independent set in the complement graph $\overline{G_{f,H}}$, so the clique cover number of $G_{f,H}$ equals the chromatic number of $\overline{G_{f,H}}$. Also note that $\overline{G_{f,H}}$ is itself a d-dependent random graph, with edge probability $q = 1 - p^2$. Therefore, by Theorem 16, $\Pr[BAD_f] < \exp(-n^{1+\varepsilon})$. Finally, let $BAD := (\bigvee_i BAD_i) \bigvee \left(\bigvee_{\text{d-limited } f} BAD_f\right)$. There are n indices i and at most $n^n \leqslant \exp(n\log n)$ d-limited functions f. Therefore, buy a union bound we have

$$\Pr[BAD] < n\Pr[BAD_i] + n^n \Pr[BAD_f] < ne^{-\frac{np^2}{2}} + n^n e^{-n^{1+\varepsilon}} < 1.$$

Therefore, there exists a good H. Also note that in $\mathcal{P}_H$ for a good H, Alice and Bob each communicate $O(n\frac{\log\log n}{\log n})$ bits. This completes the proof. □

Simultaneous Messages. We conclude this section by showing how to convert $\mathcal{P}_H$ into an SM protocol. Observe that Carol selects a bit from Alice's message (namely, the clique containing $f(i)$) and a few bits from Bob's message (the neighbors of i in H) and XORs them together. To convert $\mathcal{P}_H$ to an SM protocol, Alice and Bob send the same messages as in $\mathcal{P}_H$. Carol, given i and f, sends a bitmask describing which bit from Alice's message and which bits from Bob's message are relevant. The Referee then XORs these bits together, again producing $\text{MPJ}_3(i, f, x)$. Carol sends one bit for each bit of communication sent by Alice and Bob. Thus, this SM protocol costs twice as much as the cost of $\mathcal{P}_H$. We get the following result.

Lemma 21 (Restatement of Lemma 9). $\mathrm{D}^{\parallel}(\text{MPJ}_3) = O(n\frac{\log\log n}{\log n})$.

5 Proofs of Main Technical Lemmas

In this section, we state and prove three technical lemmas which form key insights to our contribution. The first lemma states that most sets of k vertices "look independent". The second bounds the expected number of *intersecting* k-cliques. The final lemma gives a lower bound on the expected number of *disjoint* UNCORRELATED k-cliques.

We remind the reader that all three lemmas apply to arbitrary d-dependent random graph distributions.

Lemma 22 (Restatement of Lemma 11). *Suppose d and k are integers such that $dk^3 \leqslant n$. Fix any d-dependent graph $G_d(n,p)$, and let S be a set of k vertices uniformly chosen from V. Then, we have*

$$\Pr[S \textit{ is } \text{UNCORRELATED}] \geqslant 1 - \frac{3dk^3}{2n} .$$

Proof. We divide the possible conflicts into two classes, bound the probability of each, and use a union bound. Say that correlated edges are *local* if they share a vertex. Otherwise, call them *remote*. Let $\mathcal{L}$ and $\mathcal{R}$ be the events that S contains a local and remote dependency respectively.

First, we bound $\Pr[\mathcal{R}]$. Imagine building S by picking vertices $v_1, \ldots, v_k$ one at a time uniformly. Let $S_i := \{v_1, \ldots, v_i\}$, and let B_i be the the set of vertices that would create a remote dependency if added to S_i. Note that $B_1 = \varnothing$ since there are no edges in S_1 (it contains only one vertex). More importantly, for $i > 1$, there are at most $\binom{i}{2} \cdot (2d) < di^2$ vertices in B_i, because S_i contains $\binom{i}{2}$ edges; each edge depends on at most d other edges, and each of these edges contributes at most two vertices to B_i. It follows that $\mathcal{R}$ is avoided if $v_{i+1} \notin B_i$ for each $i = 2 \ldots k-1$. There are $(n-i)$ choices for v_{i+1}, so

$$\Pr[\neg\mathcal{R}] \geqslant \prod_{i=2}^{k-1} \left(1 - \frac{di^2}{n-i}\right) \geqslant \left(1 - \frac{dk^2}{n-k}\right)^{k-2} \geqslant 1 - \frac{dk^3}{n} ,$$

Hence $\Pr[\mathcal{R}] \leqslant dk^3/n$. At first glance, it might appear like we've handled local dependencies as well. However, it is possible that when adding v_i, we add local dependent edges, if these edges are both adjacent to v_i. Thus, we handle this case separately.

Let $\mathcal{L}_{ij}$ denote the event that $i, j \in S$ and there are no local dependencies in S involving (i,j). Call a vertex ℓ bad for (i,j) if either (i,ℓ) or (j,ℓ) depend on (i,j). There are at most d bad vertices for (i,j).

Note that $\Pr[i,j \in S] = \binom{n-2}{k-2}/\binom{n}{k} = k(k-1)/n(n-1)$ and that

$$\begin{aligned}
\Pr[\neg\mathcal{L}_{ij} | i,j \in S] &\geqslant \binom{n-2-d}{k-2}/\binom{n-2}{k-2} \\
&\geqslant \prod_{z=0}^{d-1}\left(1 - \frac{k-2}{n-2-z}\right) \\
&\geqslant \left(1 - \frac{k-2}{n-2-d}\right)^d \\
&\geqslant 1 - \frac{d(k-2)}{n-2-d} \\
&\geqslant 1 - \frac{dk}{n} \,.
\end{aligned}$$

It follows that $\Pr[\mathcal{L}_{ij}] = \Pr[i,j \in S]\Pr[\mathcal{L}_{ij} | i,j \in S] \leqslant \frac{k(k-1)}{n(n-1)} \cdot \frac{dk}{n}$. There are $\binom{n}{2}$ possible pairs i,j, so by a union bound, we have $\Pr[\mathcal{L}] \leqslant \frac{n(n-1)}{2}\frac{k(k-1)}{n(n-1)}\frac{dk}{n} \leqslant \frac{dk^3}{2n}$. Another union bound on $\mathcal{R}$ and $\mathcal{L}$ completes the lemma. □

Lemma 23. *Let d, p, k be such that $k < \frac{\log(n/(2d\log^3 n))}{\log 1/p}$. Fix a d-dependent random graph distribution $G_d(n,p)$. Let $G \sim G_d(n,p)$, and let W be the set of ordered pairs (S,T) such that S,T are intersecting* UNCORRELATED *k-cliques. Then,*

$$E[|W|] \leqslant 2k\binom{n}{k}p^{2\binom{k}{2}-1}\binom{k}{2}\binom{n}{2} \,.$$

Note: To understand the relationship between d, k, p, n, it is helpful to consider the case $d = n^{o(1)}$. In this setting, the lemma holds as long as $k \leqslant (1-o(1))\frac{\log n}{\log 1/p}$.

Proof. Let S,T be arbitrary sets of k vertices, and let $X = S \cap T$. We calculate $E[|W|]$ by iterating over all possible values of S, X and for each pair, counting the expected number of T such that $S \cap T = X$ and S,T are both k-cliques. For S, X, let $F(S,X)$ be the expected number of UNCORRELATED k-cliques T such that $S \cap T = X$, conditioned on S being a k-clique. Also let $F(\ell)$ be the maximum of all $F(S,X)$, taken over all S and all $X \subset S$ with $|X| = \ell$. We have

$$\begin{aligned}
E[|W|] &= \sum_S \Pr[S \text{ is k-clique}] \sum_{X \subset S} \sum_{T : S \cap T = X} \Pr[T \text{ is k-clique} | S \text{ is k-clique}] && (1) \\
&= \sum_S p^{\binom{k}{2}} \sum_{X \subset S} F(S,X) && (2) \\
&\leqslant \sum_S p^{\binom{k}{2}} \sum_{\ell=2}^{k-1} \sum_{\substack{X \subset S \\ |X| = \ell}} F(\ell) && (3) \\
&\leqslant \binom{n}{k} p^{\binom{k}{2}} \sum_\ell \binom{k}{\ell} F(\ell) \,. && (4)
\end{aligned}$$

Next, we obtain an upper bound on $F(\ell)$. Since we need only an upper bound, we take a very pessimistic approach. Let $M \subset [n] \setminus S$ be the set of vertices adjacent to an edge e that depends on some edge from

$S\backslash X$. Each edge in $S\backslash X$ depends on at most d other edges, and there are $\binom{k}{2} - \binom{\ell}{2}$ edges in $S\backslash X$. Therefore, $|M| \leqslant d(\binom{k}{2} - \binom{\ell}{2})$. Now, let $E(M)$ be the set of edges with one endpoint in M and the other endpoint in $M \cup X$. Each of these edges may be correlated with edges in $S\backslash X$, so for any $e \in E(M)$ we assume only $\Pr[e|S \text{ is k-clique}] \leqslant 1$. On the other hand, by construction any edge e *not* in $E(M)$ is independent of S, and therefore $\Pr[e \in G|S \text{ is k-clique}] = p$. Next, we sum over all possible T, grouping by how much T instersects M. Suppose $|T \cap M| = \ell'$ for some $0 \leqslant \ell' \leqslant k - \ell$. Then, T contains $\binom{k}{2}$ edges, $\binom{\ell}{2}$ of these edges have both endpoints in X, and are fixed after conditioning on S being a k-clique. Of the remaining edges, $\ell \cdot \ell' + \binom{\ell'}{2}$ are in $E(M)$; the rest are independent of S. Thus, when $|T \cap M| = \ell'$, then $\Pr[T \text{ is k-clique}|S \text{ is k-clique}] \leqslant p^{\binom{k}{2} - \binom{\ell}{2} - \ell\ell' - \binom{\ell'}{2}}$.

$$
\begin{aligned}
F(\ell) &= \sum_{T : S \cap T = X} \Pr[T \text{ is k-clique}|S \text{ is k-clique}] && (5)\\
&= \sum_{\ell'=0}^{k-\ell} \sum_{\substack{T : S \cap T = X \\ |T \cap M| = \ell'}} \Pr[T \text{ is k-clique}|S \text{ is k-clique}] && (6)\\
&\leqslant \sum_{\ell'=0}^{k-\ell} \binom{M}{\ell'} \binom{n-k-M}{k-\ell-\ell'} p^{\binom{k}{2} - \binom{\ell}{2} - \ell\ell' - \binom{\ell'}{2}} && (7)\\
&= p^{\binom{k}{2} - \binom{\ell}{2}} \sum_{\ell'=0}^{k-\ell} F^*(\ell') \,, && (8)
\end{aligned}
$$

where $F^*(\ell') := \binom{M}{\ell'}\binom{n-k-M}{k-\ell-\ell'} p^{-\ell\ell' - \binom{\ell'}{2}}$. Next, we show that the summation in Equation (8) telescopes.

Claim 24. *If* $k \leqslant \frac{\log\left(\frac{n}{2d \log^3 n}\right)}{\log 1/p}$ *then* $\sum_{\ell'=0}^{k-1} F^*(\ell') \leqslant 2F^*(0)$.

Proof. Fix any $0 \leqslant i < k - \ell$, and consider $F^*(i+1)/F^*(i)$. Using $\binom{a}{b+1}/\binom{a}{b} = \frac{a-b}{b+1}$ and $\binom{a}{b-1}/\binom{a}{b} = \frac{b}{a-b-1}$ and recalling that $M < d\binom{k}{2}$, we have:

$$
\begin{aligned}
\frac{F^*(i+1)}{F^*(i)} &= \frac{\binom{M}{i+1}\binom{n-k-M}{k-(i+1)} p^{-\ell(i+1) - (i+1)i/2}}{\binom{M}{i}\binom{n-k-M}{k-i} p^{-\ell i - i(i-1)/2}} \\
&= \frac{M-1}{i+1} \frac{k-i}{n-k-M-k+i} p^{-\ell-i} \\
&\leqslant \frac{dk^2}{2} \frac{k}{n - o(n)} \left(\frac{1}{p}\right)^k \\
&< \frac{dk^3}{n} \left(\frac{1}{p}\right)^k \\
&< \frac{k^3}{2\log^3 n} \\
&< 1/2 \,,
\end{aligned}
$$

where the penultimate inequality holds because of our assumption on k, and the final inequality holds because $k < \log n$. We've shown that for all i, $F^*(i+1)/F^*(i) < 1/2$. Hence $F^*(i) < F^*(0)2^{-i}$, and so $\sum_{\ell'} F^*(\ell') \leqslant \sum_{\ell'} F^*(0)2^{-\ell'} \leqslant 2F^*(0)$. □

From claim 24, we see that

$$F(\ell) \leqslant p^{\binom{k}{2}-\binom{\ell}{2}} \sum_{\ell'=0}^{k-\ell} F^*(\ell') \leqslant 2p^{\binom{k}{2}-\binom{\ell}{2}} F^*(0) = 2p^{\binom{k}{2}-\binom{\ell}{2}} \binom{n-k-M}{k-\ell} .$$

Now, plugging this inequality back into Equation 4, we get

$$E[|W|] \leqslant \binom{n}{k} p^{\binom{k}{2}} \sum_{\ell} \binom{k}{\ell} F(\ell) \leqslant 2\binom{n}{k} p^{\binom{k}{2}} \sum_{\ell} \binom{k}{\ell} p^{\binom{k}{2}-\binom{\ell}{2}} \binom{n-k-M}{k-\ell} .$$

Let $G(\ell) := p^{\binom{k}{2}-\binom{\ell}{2}} \binom{k}{\ell} \binom{n-k-M}{k-\ell}$, and for $2 \leqslant \ell < k-1$, let $G^*(\ell) := G(\ell)/G(\ell+1)$. Note that

$$G^*(\ell) = p^{\ell} \frac{\ell+1}{k-\ell} \frac{n-2k-M+\ell+1}{k-\ell} .$$

We claim that $G^*(\ell)$ decreases as long as $p < 8/27 - \Omega(1)$. To see this, note that

$$\frac{G^*(\ell)}{G^*(\ell+1)} = p\frac{\ell+1}{\ell} \cdot \left(\frac{k-\ell+1}{k-\ell}\right)^2 \frac{n-2k-M+\ell+1}{n-2k-M+\ell} < p(3/2)^3(1+o(1)) ,$$

where the inequality holds because $(a+1)/a = 1 + 1/a$ and because $\ell, k-\ell \geqslant 2$ for the range of ℓ we need when calculating $G^*(\ell)$. In a way, saying that $G^*(\ell)$ is decreasing amounts to saying that $G(\ell)$ is convex—once $G(i) \leqslant G(i+1)$, then $G(j) \leqslant G(j+1)$ for all $j > i$. Next, a straightforward calculation using our choice of k shows that $G(k-1) \leqslant G(2)$. Thus, it must be the case that $G(i) \leqslant G(2)$ for all i, and therefore

$$E[|W|] \leqslant 2\binom{n}{k} p^{\binom{k}{2}} kG(2) = 2k\binom{n}{k} p^{2\binom{k}{2}-1} \binom{k}{2} \binom{n-k-M}{k-2} < 2k\binom{n}{k} p^{2\binom{k}{2}-1} \binom{k}{2} \binom{n}{k-2} .$$

This completes the proof of Lemma 23. □

Finally, we prove the lemma that in any d-dependent graph distribution, the expected number of *disjoint* UNCORRELATED k-cliques is large. Recall that Y is the maximal number of disjoint UNCORRELATED k-cliques.

Lemma 25 (Restatement of Lemma 15). $\mathbb{E}[Y] \geqslant \frac{n^2 p}{19k^5}$.

Proof. We construct Y probabilistically, by selecting each potential UNCORRELATED k-clique with small probability and removing any pairs of k-cliques that intersect. Let K denote the family of UNCORRELATED k-cliques. By Lemma 11 and our choice of d, a randomly chosen set S of k vertices is UNCORRELATED with probability at least $2/3$. By this and our choice of k, we have

$$\mathbb{E}[|K|] \geqslant \frac{2}{3}\binom{n}{k} p^{\binom{k}{2}} .$$

Recall that W is the set of ordered pairs $\{S, T\}$ of UNCORRELATED k-cliques such that $2 \leqslant |S \cap T| < k$. For our argument, we require an upper bound on $\mathbb{E}[|W|]$. In the standard random graph model, if $|S \cap T| = \ell$, then $\Pr[S, T \text{ both k-cliques}] = p^{\binom{k}{2}-\binom{\ell}{2}}$. However, this no longer holds for d-dependent distrubtions, even if S and T are both UNCORRELATED. This is because while edges in S and T are independent,

edges in S but not T may be correlated with edges in T but not S. As an extreme case, suppose all edges in S are independent, but each edge in $S \setminus T$ is completely correlated with an edge in $T \setminus S$. Then, $\Pr[S, T \text{ k-cliques}] = \Pr[S \text{ is k-clique}] = \Pr[T \text{ is k-clique}] = p^{\binom{k}{2}}$. Essentially, allowing edges to be correlated has the potential to drive up the variance on the number of k-cliques, even when these k-cliques are UNCORRELATED. This is perhaps to be expected. Nevertheless, in Lemma 23, we were able to show that when d is small, this increase is not much more than in the standard graph model.

With this claim, we are now able to construct a large set of disjoint UNCORRELATED k-cliques with high probability. Create $K' \subseteq K$ by selecting each uncorrelated $S \in K$ independently with probability

$$\Pr[S \in K'] = \gamma = \frac{1}{12kp^{\binom{k}{2}-1}\binom{k}{2}\binom{n}{k-2}} .$$

Finally, create L from K' by removing each pair $S, T \in K'$ such that $S, T \in W$. By construction, L is a set of edge-disjoint UNCORRELATED k-cliques; furthermore, we have

$$\begin{aligned}
E[|L|] &= \gamma E[|K|] - 2\gamma^2 E[|W|] \\
&\geqslant \frac{2\gamma}{3}\binom{n}{k}p^{\binom{k}{2}} - \frac{2\gamma \cdot 2k\binom{n}{k}p^{2\binom{k}{2}-1}\binom{k}{2}\binom{n}{k-2}}{12kp^{\binom{k}{2}-1}\binom{k}{2}\binom{n}{k-2}} \\
&= \frac{2\gamma}{3}\binom{n}{k}p^{\binom{k}{2}} - \frac{\gamma}{3}\binom{n}{k}p^{\binom{k}{2}} \\
&= \frac{\gamma}{3}\binom{n}{k}p^{\binom{k}{2}} \\
&= \frac{\binom{n}{k}p^{\binom{k}{2}}}{3 \cdot 12kp^{\binom{k}{2}-1}\binom{k}{2}\binom{n}{2}} \\
&\geqslant \frac{\binom{n}{k}}{\binom{n}{k-2}}\frac{p}{36k}\frac{1}{\binom{k}{2}} \\
&\geqslant \frac{p}{18k^3}\frac{\binom{n}{k}}{\binom{n}{k-2}} \\
&= \frac{p}{18k^3}\frac{(n-k-2)(n-k-1)}{k(k-1)} \\
&\geqslant \frac{p}{18k^3}\frac{18n^2}{19k^2} \\
&= \frac{n^2 p}{19k^5} ,
\end{aligned}$$

where the final inequality holds for large enough n. □

6 Results for Non-Boolean Pointer Jumping

In this section, we leverage the protocol for MPJ_3 to achieve new results for the non-Boolean Pointer Jumping problem $\widehat{\text{MPJ}}$. Let $\mathcal{Q}$ be the protocol for MPJ_3 given in Lemma 9. First, we give a protocol for $\widehat{\text{MPJ}}_3$. The cost matches the upper bound from [10] but has the advantange of working in the Simultaneous Messages model.

Lemma 26 (Restatement of Lemma 10). *There is an $O(n \log\log n)$-bit SM protocol for $\widehat{\text{MPJ}}_3$.*

Proof. Run $\mathcal{Q} \log n$ times in parallel, on inputs $(i, f_2, z_1), (i, f_2, z_2), \ldots, (i, f_2, z_{\log n})$, where z_j denotes the jth most significant bit of f_3. This allows the Referee to recover each bit of $f_3(f_2(i)) = \widehat{\text{MPJ}}(i, f_2, f_3)$. □

Next we give a new upper bound for $\widehat{\text{MPJ}}_4$. As far as we know, this is the first protocol for $\widehat{\text{MPJ}}_k$ for any k that uses a sublinear amount of communication.

Theorem 27 (Restatement of Theorem 8). *There is a one-way protocol for $\widehat{\text{MPJ}}_4$ with cost $O(n\frac{(\log\log n)^2}{\log n})$.*

Proof. Let i, f_2, f_3, f_4 be the inputs to $\widehat{\text{MPJ}}_4$, and for $1 \leqslant j \leqslant \log n$, let $z_j \in \{0,1\}^n$ be the string obtained by taking the jth most significant bit of each $f_3(w)$ (i.e., $z_j[w]$ is the jth most significant bit of $f_3(w)$.) Fix a parameter k to be determined shortly. $\text{PLR}_1, \text{PLR}_2$, and PLR_3 run $\mathcal{Q}$ on $\{(i, f_2, z_j) : 1 \leqslant j \leqslant k\}$. From this, PLR_3 learns the first k bits of $f_3(f_2(i))$. She then sends $f_4(z)$ for every $z \in \{0,1\}^{\log n}$ whose k most significant bits match those of $f_3(f_2(i))$. PLR_4 sees i, f_2, and f_3, computes $z^* := f_3(f_2(i))$, and recovers $f_4(z^*)$ from PLR_3's message. Note that there are $n/2^k$ strings that agree on the first k bits, and for each of these strings, PLR_3 sends $\log n$ bits. Therefore, the cost of this protocol is $k\,\text{cost}(\mathcal{Q}) + n\log(n)/2^k = O\left(kn\frac{\log\log n}{\log n} + n\log(n)2^{-k}\right)$. Setting $k := 2\log\frac{\ln 2 \log n}{\log\log n} = \Theta(\log\log n)$ minimizes the communication cost, giving a protocol with cost $O\left(n\frac{(\log\log n)^2}{\log n}\right)$. □

7 Dependent Graphs with Large Cliques or Large Dependency

In this section, we provide results that witness the tightness of our current bounds. The next lemma shows that there exist dependent random graphs that almost surely contain cliques of size $\Omega(d)$, and others that almost surely have cliques of size $\Omega(\sqrt{d}\log(n))$.

Lemma 28. *(Restatement of Lemma 5) For all constant $0 < p < 1$ and $d = o(n)$,*

1. *there exists a d-dependent random graph $G_d(n,p)$ such that*

$$\Pr\left[\text{clique}(G_d(n,p)) > \frac{d\sqrt{p}}{2} - d^{\frac{1}{2}}p^{\frac{1}{4}}\right] > 1 - e^{-2n/d}\,.$$

2. *there exists a d-dependent random graph $G_d(n,p)$ such that almost surely*

$$\text{clique}(G_d(n,p)) = \Omega(\sqrt{d}\log(n))\,.$$

Proof. We give two constructions.

For the first result, fix $d' := \frac{d\sqrt{p}}{2} - \sqrt{d\sqrt{p}}$ and $M_1 := 2n/d$. Partition the vertices into M_1 sets $V_1, \ldots, V_{M_1}$ each of size $d/2$. Let $c(i)$ denote the part containing i (we think of i has having color c). Now, let $\{X_{i,c} : i \in V, 1 \leqslant c \leqslant M_1\}$ be a series of i.i.d. random bits with $\Pr[X_{i,c} = 1] = \sqrt{p}$, and place $(i,j) \in G_d(n,p)$ if $X_{i,c(j)} \bigwedge X_{j,c(i)} = 1$. Thus, (i,j) is an edge with probability p. Also note that edges (i,j) and (i',j') are dependent if either $c(i) = c(i')$ or $c(j) = c(j')$. Since there are $d/2$ vertices in each V_ℓ, (i,j) is dependent on at most d other edges and $G_d(n,p)$ is d-dependent.

Now, fix a color c, and let $S_c := \{i : c(i) = c \wedge X_{i,c} = 1\}$. For any $i, j \in S_c$ we have $X_{i,c} = X_{j,c} = 1$ and that $c(i) = c(j) = c$. Therefore, $(i,j) \in G_d(n,p)$ for any $i, j \in S_c$, hence S_c is a clique.

Next, consider $|S_c|$. There are $d/2$ vertices with color c, so $E[|S_c|] = \frac{d\sqrt{p}}{2}$. By the Chernoff bound, $\Pr[|S_c| < d'] < \frac{1}{e}$, so the probability that there is some color c with $|S_c| \geqslant d'$ is at least $1 - e^{-2n/d}$. Therefore, $G_d(n,p)$ almost surely contains a clique of size at least d'.

For the second graph, partition the vertices $[n]$ into $M_2 := n/\sqrt{d}$ subsets $V_1, \ldots, V_{M_2}$, each of size $\sqrt{d}$. Let $c(i)$ be the subset containing i. Let $\{X_{c_1,c_2} : 1 \leqslant c_1, c_2 \leqslant M_2\}$ be a set of independent, identically distributed binary variables with $\Pr[X_{c,c'} = 1] = p$. Now, place edge (i,j) in the graph if $X_{c(i),c(j)} = 1$. In this way, for any V_s, V_t, either all edges between V_s and V_t exist, or none do, and similarly for any V_s, either all edges between vertices in V_s will be in the graph, or none will.

Next, let S be the set of all i such that edges between vertices in V_i are in the graph. Each $i \in S$ with probability p. By standard Chernoff bounds, $|S| \geqslant pM_2/2$ with high probability. Let $M' := pM_2/2$. The construction above induces a new random graph G' on M' vertices where all edges are i.i.d. in G' with probablity p. i.e., G' is an Erdős-Rényi random graph on M' vertices. By [7], $\mathrm{clique}(G') \geqslant 2\log(M')/\log(1/p) = \Omega(\log(n)/\log(1/p))$ with high probability. Finally, a clique of size k in G' gives a clique of size $k\sqrt{d}$ in G, hence G contains a clique of size $\Omega(\sqrt{d}\log(n)/\log(1/p))$ with high probablity. □

Our second result in this section shows that when the dependency factor becomes $\Omega(n)$, essentially nothing can be said about the clique number of dependent random graphs.

Lemma 29. *(Restatement of Lemma 6) Fix $d := 2n - 2$. Then, the following statements hold.*

1. *For any $0 < p < 1$, there exists a d-dependent random graph $G_d(n,p)$ that is bipartite with certainty.*
2. *For any $1/2 \leqslant p < 1$, there exists a d-dependent random graph $G_d(n,p)$ such that $\mathrm{clique}(G_d(n,p)) \geqslant n/2$ with certainty.*

Proof. We again provide two constructions. For the first construction, set $q_1 := 1 - \sqrt{1-p}$, and let $X_1, \ldots, X_n$ be i.i.d. random bits such that $X_i = 1$ with probability q_1. Think of each X_i as being assinged to vertex v_i. Now, place edge $(i,j) \in G_d(n,p)$ iff $X_i \oplus X_j = 1$. Note that $(i,j) \in G_d(n,p)$ with probability $2q(1-q) = p$. It is easy to see that (i,j) depends on (i',j') only if either $i = i'$ or $j = j'$. There are at most $2(n-1)$ such edges, hence the random graph is d-dependent. Finally, we claim that the graph is bipartite. To see this, suppose for the sake of contradiction that $G_d(n,p)$ contains an odd cycle $(1, 2, \ldots, 2k+1, 1)$. Without loss of generality, assume that $X_1 = 1$ (the proof is similar if $X_1 = 0$.) Since each edge $(i, i+1) \in G_d(n,p)$, we must have that $X_2, X_4, \ldots, X_{2k}$ all equal 0, and $X_1, X_3, \ldots, X_{2k+1}$ all equal 1. But then $X_1 = X_{2k+1} = 1$, hence $(1, 2k+1) \notin G_d(n,p)$. This contradicts the assumption that $(1, 2, \ldots, 2k+1, 1)$ is a cycle.

We proceed with the second construction in a similar manner. Let $q_2 := \frac{1}{2}(1 - \sqrt{2p-1})$, and let $X_1, \ldots, X_n$ be i.i.d. random bits with $\Pr[X_i = 1] = q_2$. This time, place $(i,j) \in G_d(n,p)$ iff $X_i = X_j$. Note that (i,j) is an edge with probablity $q_2^2 + (1-q_2)^2 = p$. Now, let $S_0 := \{i : X_i = 0\}$ and similarly $S_1 := \{i : X_i = 1\}$. It is easy to see that S_0 and S_1 are both cliques in $G_d(n,p)$. One of them must contain at least half the vertices. □

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